

The Stochastic Goodwill Problem

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August 10, 2005

Abstract

Stochastic control problems related to optimal advertising under uncertainty are considered. In particular, we determine the optimal strategies for the problem of maximizing the utility of goodwill at launch time and minimizing the disutility of a stream of advertising costs that extends until the launch time for some classes of stochastic perturbations of the classical Nerlove-Arrow dynamics. We also consider some generalizations such as problems with constrained budget and with discretionary launching.

Key Words: optimal advertising under uncertainty, Bellman equation, control with discretionary stopping.

1 Introduction

We consider the optimization problem faced by a firm that, while advertising a product prior to its introduction to the market, wants to determine the optimal advertising policy for the maximization of the product image (also called *goodwill*), and the minimization of the total discounted cost. We shall also consider the problem of optimizing the launching time, thus allowing the firm to decide at its discretion to stop the advertising campaign and start selling the product.

This type of problems can be traced back at least to Nerlove and Arrow [22], who proposed to model the stock of advertising goodwill $x(t)$ at time $t \geq 0$ as

$$\dot{x}_t = u_t - \rho x_t, \quad x_0 = x \geq 0, \quad (1)$$

where u_t is the rate of advertising expenditure, $\rho > 0$ is a factor of deterioration of product image in absence of advertisement. The optimization problem for a firm that seeks to maximize awareness of its product at a given launch time $T > 0$ and to minimize its advertising effort until T could be formulated as a multi-objective program of the type

$$\left(\sup_{u \in \mathcal{U}} \mathbb{E}[e^{-cT} \varphi(x_T)], \inf_{u \in \mathcal{U}} \mathbb{E} \left[\int_0^T e^{-ct} h(u_t) dt \right] \right), \quad (2)$$

subject to the dynamics (1), where $c > 0$ is a discount factor, $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a reward function, $h : \mathbb{R} \rightarrow \mathbb{R}$ is a loss function, and \mathcal{U} is the set of measurable functions

$u : [0, T] \rightarrow U$, with U a closed subset of \mathbb{R}_+ . Following a standard procedure in multi-objective optimization (see e.g. Zeleny [28]), one takes a weighted average of the objectives in (2) and obtains the deterministic optimal control problem

$$\sup_{u \in \mathcal{U}} \left(e^{-cT} \gamma_0 \varphi(x_T) - \int_0^T e^{-ct} h(u_t) dt \right), \quad (3)$$

where γ_0 is a positive constant. Sufficient conditions for the problem to be well posed are, e.g., that φ is concave and continuous, h is convex and continuous, and U is compact. The conditions are also meaningful from an economic point of view, as it is customary to use concave (increasing) utility functions as measures of reward, and convex (increasing) loss functions. In the simplest case, one could take $h(u) = u$, so that the second term in (3) coincides with the total discounted advertising expenditure. This deterministic optimal control formulation has been extended by many authors to account for delay effects, non-linearity in the response to advertisement, and many other factors. For a recent work on the subject, which also contains a list of related references, we refer to Buratto and Viscolani [7].

On the other hand, less work has been devoted to the case of stochastic evolution of goodwill level: for a few examples of works in this direction, we refer to the survey by Feichtinger, Hartl and Sethi [11] and references therein, and to the more recent papers of Grosset and Viscolani [13] and Buratto and Grosset [6]. The emergence of randomness in the dynamics of goodwill is quite natural for several reasons: one may think, for example, that random fluctuations in the goodwill level are the effect of external factors beyond the control of the firm, or that noise enters through the control, since the effect of advertising may be partly uncertain (see section 2 for a detailed discussion).

In this work we introduce some stochastic extensions of the classical model of Nerlove and Arrow and study related optimization problems. We do not aim at maximum generality, instead we focus on models whose special structure allows us to obtain explicit solutions. In particular, in section 2 we propose a stochastic extension of the Nerlove-Arrow dynamics and motivate it by marketing assumptions. We formulate a rather general problem of optimizing an objective function that weighs (a function of) product image at a fixed time and the cumulative cost of advertising effort, and we construct a nearly optimal advertising strategy. The special case of linear reward of goodwill and linear cost of advertising effort is considered in section 3: the special structure of the problem allows one to obtain the value function and the optimal policy in closed form, and to consider more general problems of advertising with a limited budget. In section 4 we study another case where explicit solutions can be obtained, i.e. the case of quadratic reward of final goodwill and quadratic cost of advertising. Under these assumptions we also explicitly solve in section 5 a problem of optimal advertising with discretionary stopping to reach a target level of product awareness. We conclude suggesting some problems not addressed in this paper.

2 Stochastic models for goodwill dynamics and related optimization problems

Let x_t be the level of product image at time t , $0 \leq t \leq T$, where $T > 0$ is the end of the planning horizon (the time at which the product will be launched). We postulate

a dynamics for x_t given by the following stochastic differential equation, which is a stochastic perturbation of the Nerlove-Arrow dynamics (1):

$$dx_t = (-\rho x_t + u_t) dt + \sigma(x_t, u_t) dw_t, \quad x_0 = x \geq 0, \quad (4)$$

where ρ is a positive constant, $\sigma : \mathbb{R}^2 \rightarrow \mathbb{R}$ is Lipschitz continuous, and w_t is a standard real valued Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}$. The control process u_t models the rate of advertising *effort* by the firm, and is assumed to be measurable, adapted, and taking values in a closed convex subset U of $[0, +\infty)$. We will denote by \mathcal{U} the set of controls satisfying these properties. We use gross rating points (GRPs) to measure advertising effort, instead of the rate of advertising expenditure, following a recent trend in the marketing literature – see e.g. Dube and Manchanda [8], Vilcassim, Kadiyali and Chintagunta [27].

As briefly mentioned in the introduction, there are several reasons to study stochastic extensions of the Nerlove-Arrow model. Early papers on the subject such as Rao [25] and Raman [24] advocated the use of stochastic models with the observation that such effects as copy and competitive changes would render uncertain the effect of advertising on goodwill. In both papers stochastic perturbations of the Nerlove-Arrow dynamics of the type (4) were proposed, with σ not depending on x_t nor u_t .

While a model with only additive noise can be useful as a first approximation, it is reasonable to consider more general stochastic disturbances, depending also on the goodwill level x_t and on the intensity of advertising effort u_t . In particular, one could distinguish, potentially among others, three sources of uncertainty: a “background noise”, of additive type, due to the kind of effects indicated by Raman [24] and Rao [25], which are not directly influenced neither by the popularity of the product nor by the intensity of advertising. A second contribution to the intensity of noise can be attributed to the uncertainty in the opinion of the (potential) customers that are aware of the product. Finally, one should take into account the uncertainty in the effect of advertising. In order to model explicitly such disturbances, it may be reasonable to assume that the intensities of the second and third type of noise just mentioned are proportional respectively to the goodwill level x_t (a proxy for the number of customers that are aware of the advertised product) and to the level of advertising effort u_t . In particular, the noise component proportional to the goodwill level could also be interpreted as the effect of “internal influence” (also called word-of-mouth communication), due to the random outcome on the goodwill level of the interaction between customers who know the product. Similar ideas (and terminology) are extensively employed in the literature on new product diffusion (see e.g. Bass [2]). The third source of uncertainty described above could be suggestively justified by attaching to each unit of advertising effort a random effect, so that, heuristically speaking, “noise enters the system through the control” via $u_t \mapsto u_t(1 + \sigma_2 \frac{dw_t}{dt})$.

We would also like to mention that models for the evolution of goodwill expressed as stochastic perturbations of the Nerlove-Arrow dynamics of the type (4), where the diffusion coefficient σ depends both on x_t and on u_t , appeared also as diffusion approximations of models based on discrete-time Markov chains (see e.g. Tapiero [26] and references therein), and in the above mentioned papers [13], [6].

Due to the lack of empirical studies and of theoretical papers in the marketing literature on the determinants of uncertainty in the dynamics of goodwill, we are led to

consider, in the same spirit of the cited works of Raman and Rao, a linear specification of the diffusion coefficient σ :

$$\sigma(x, u) = \sigma_0 + \sigma_1|x| + \sigma_2 u,$$

where $\sigma_0, \sigma_1, \sigma_2$ are fixed non-negative constants. In practice, the values of these coefficients should be determined by ad hoc empirical studies and/or by specific managerial and marketing insights.

A natural analog in the stochastic setting of the general optimization problem (3) can now be formulated. Let us define the performance functional relative to strategy $u \in \mathfrak{U}$ as

$$v^u(s, x) = \mathbb{E}_{s,x}^u \left[e^{-cT} \varphi(x_T) - \int_0^T e^{-ct} h(u_t) dt \right], \quad (5)$$

and the value function as

$$v(s, x) = \sup_{u \in \mathfrak{U}} v^u(s, x), \quad (6)$$

where $h : U \rightarrow \mathbb{R}_+$ is bounded and $|\varphi(x)| < K(1 + |x|^m)$ for some positive constants K, m . By $\mathbb{E}_{s,x}^u$ we mean, as usual, expectation with respect to the law of the controlled diffusion

$$x_t = x + \int_s^t (-\rho x_r + u_r) dr + \int_s^t (\sigma_0 + \sigma_1|x_r| + \sigma_2 u_r) dw_r, \quad s \leq r \leq t \leq T.$$

Our objective is to characterize the value function and to find (or approximate) strategies realizing the supremum in (6). Although the problem is well posed under the given assumptions, particularly meaningful choices from the economic point of view are φ concave increasing (we have in mind the utility function of a risk-averse agent) and h convex increasing (typical choice of cost function).

We shall study the problem through the dynamic programming approach, i.e. through the study of the associated Bellman equation, which can be written as

$$\sup_{u \in U} \left[\frac{\partial \psi}{\partial t} + L^u \psi - c\psi - h(u) \right] = 0, \quad \psi(T, x) = \varphi(x), \quad (7)$$

where L^u is the differential operator defined by

$$L^u = \frac{1}{2} a(x, u) \partial_x^2 + b(x, u) \partial_x,$$

with $a := \sigma \sigma^* = \sigma^2$ and $b(x, u) = -\rho x + u$.

Equation (7) is (under the assumption $\sigma_0 \neq 0$) a fully nonlinear uniformly nondegenerate parabolic PDE, for which general results about existence of smooth solutions are available under additional assumptions of smoothness and boundedness of the coefficients and of the reward and loss functions. However, we can prove existence of nearly optimal control in a general setting that covers also the case $\sigma_0 = 0$, for which the Bellman equation (7) becomes degenerate. In particular we have the following result.

Theorem 1 *For any $\varepsilon > 0$, $s \in [0, T]$, $x \in \mathbb{R}$, there exists a Markov strategy $u_t^\varepsilon \in \mathfrak{U}$ such that $v(s, x) \leq v^{u^\varepsilon}(s, x) + \frac{4}{3}\varepsilon$.*

Proof. We divide the proof in three steps. In the first step we introduce a sequence of approximating problems, in the second step we construct an optimal feedback control for each approximating problem. In the last step we prove that optimal controls for the approximating problems are nearly optimal for the original problem.

STEP 1: Let $\zeta_n \in C_0^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \zeta_n(x) dx = 1$, $n = 0, 1, 2, \dots$, be a sequence of mollifiers. Define

$$\tilde{b}_n(x, u) = \begin{cases} \rho n + u, & x < -n \\ -\rho x + u, & -n \leq x \leq n \\ -\rho n + u, & x > n \end{cases}$$

and $b_n(x, u) = \tilde{b}_n(x, u) * \zeta_n(x)$ (convolution with respect to x). Similarly, define

$$\tilde{\sigma}_n(x, u) = \begin{cases} \sigma_0 + \sigma_1 1/n + \sigma_2 u, & |x| < 1/n \\ \sigma_0 + \sigma_1 |x| + \sigma_2 u, & 1/n \leq |x| \leq n \\ \sigma_0 + \sigma_1 n + \sigma_2 u, & |x| > n, \end{cases}$$

$$\tilde{\varphi}_n(x) = \begin{cases} \varphi(-n), & x < -n \\ \varphi(x), & -n \leq x \leq n \\ \varphi(n), & x > n, \end{cases}$$

and $\sigma_n(x, u) = \tilde{\sigma}_n(x, u) * \zeta_n(x)$, $\varphi_n(x) = \tilde{\varphi}_n(x) * \zeta_n(x)$. Similarly, let $h_n(x) = h(x) * \zeta_n(x)$. Given a sequence of functions $f_n(x, u)$, we shall say that $f_n(x, u)$ converges to $f(x, u)$ in \mathcal{L} if for each $R > 0$ one has

$$\lim_{n \rightarrow \infty} \sup_{u \in U} \sup_{|x| \leq R} |f_n(x, u) - f(x, u)| = 0.$$

By well known properties of convolution with smooth kernels, it is easy to prove that σ_n , b_n , h_n , φ_n belong to C_b^3 and that they converge to σ , b , h , φ , respectively, in \mathcal{L} . Let us denote by $x_t^{u, s, x}(n)$ a solution of the equation

$$x_t = x + \int_s^t b_n(u_r, x_r) dr + \int_s^t \sigma_n(u_r, x_r) dw_r,$$

where $u \in \mathfrak{U}$, $0 \leq s \leq t \leq T$, $x \in \mathbb{R}_+$. Moreover, let us define

$$v_n^u(s, x) := \mathbb{E}_{s, x}^u \left[- \int_s^T e^{-cr} h_n(u_r) dr + e^{-cT} \varphi_n(x_T(n)) \right] \quad (8)$$

and $v_n(s, x) = \sup_{u \in \mathfrak{U}} v_n^u(s, x)$.

STEP 2: Since $\tilde{\sigma}_n(x, u)$ is continuous and $\tilde{\sigma}_n^2(x, u) > 0$ for all $x \in \mathbb{R}$ and $u \in U$, by properties of convolutions with smooth kernels, one also has $\sigma_n^2(x, u) > 0$ uniformly over x, u for n large enough. Therefore (for a fixed large n) the value function v_n is a $C^{1,2}([0, T] \times \mathbb{R})$ solution of the Bellman equation

$$\begin{cases} \sup_{u \in U} \left[v_t(t, x) + \frac{1}{2} \sigma_n^2(x, u) v_{xx}(t, x) + b_n(x, u) v_x(t, x) - cv(t, x) - h_n(u) \right] = 0 \\ v(T, x) = \varphi_n(x), \end{cases} \quad (9)$$

as it follows from a result of Krylov (see [17], p. 301 and also [12], p. 168). As it is well known, if the supremum in (9) is attained for each (x, t) by $u^n(x, t)$, then $u_t^n := u^n(x_t, t)$

is an optimal Markov strategy for the approximating problem of maximizing (8). See also [12], p. 169.

STEP 3: As it follows by theorem 3.1.12 in [16], we have that $v_n^u(t, x) \rightarrow v^u(t, x)$ uniformly with respect to $u \in \mathfrak{U}$, $s \in [0, T]$, $|x| \leq R$ for each $R > 0$. Therefore, given $\varepsilon > 0$, there exists $N > 0$ such that for all $n > N$ one has $|v_n^u - v^u| < \varepsilon/3$ for all $u \in \mathfrak{U}$. Let us first prove that $|v_n - v| < \varepsilon$. Assume, by contradiction, that $v_n < v - \varepsilon$ (without loss of generality, as the case $v < v_n - \varepsilon$ is completely analogous). By definition of v , there exists u^1 such that $|v^{u^1} - v| < \varepsilon/3$. From $|v_n^{u^1} - v^{u^1}| < \varepsilon/3$ we also have

$$|v_n^{u^1} - v| \leq |v_n^{u^1} - v^{u^1}| + |v^{u^1} - v| < \frac{2}{3}\varepsilon,$$

hence $v_n^{u^1} > v_n$, which is absurd. Then we have proved that $|v_n - v| < \varepsilon$. Let us now take $n > N$ large enough, so that the conditions of step 2 are satisfied, and let u^ε be a Markov strategy such that $v_n = v_n^{u^\varepsilon}$. Then we have, by the triangular inequality,

$$\begin{aligned} |v^{u^\varepsilon} - v| &\leq |v^{u^\varepsilon} - v_n^{u^\varepsilon}| + |v_n^{u^\varepsilon} - v| \\ &< \frac{\varepsilon}{3} + \varepsilon = \frac{4}{3}\varepsilon, \end{aligned}$$

which proves the claim. The proof of the theorem is thus finished. \square

Under more specific structural assumptions on the objective function to optimize, it is natural to expect sharper characterizations of the value function and of the optimal advertising strategy. This is the topic of the following sections.

3 Linear reward and loss functions

In this section we study the simplest case, with linear reward for the level of goodwill at time T and linear loss function of advertising effort. As discussed before, if we identify u_t with the rate of advertising spending, then the second term in the objective function (5) can be identified with discounted cumulative advertising costs. Let us specify the problem in detail. Our aim is to maximize over \mathfrak{U} the functional

$$v^u(s, x) = \mathbb{E}_{s,x}^u \left[\gamma_0 e^{-cT} x_T - \int_0^T e^{-ct} u_t dt, \right]$$

where

$$x_t = x + \int_s^t (-\rho x_r + u_r) dr + \int_s^t \sigma(x_r, u_r) dw_r, \quad s \leq r \leq t \leq T. \quad (10)$$

In the sequel we shall set, for simplicity, $\gamma = \gamma_0 e^{-cT}$. Note that, due to the linearity on u_t of the performance functional, we can explicitly solve the optimization problem even without specifying the functional form of the diffusion coefficient σ , as long as (10) admits a solution. In particular, σ could be identically zero, for which we obtain the classical deterministic Nerlove-Arrow dynamics. The interpretation of this fact is simply that optimizing a linear objective function as v^u simply coincides with controlling

the “mean evolution” of our stochastic dynamics, for which we can obtain an explicit expression as follows: first write

$$x_T = e^{-\rho T} x + \int_0^T e^{-\rho(T-t)} u_t dt + \int_0^T e^{-\rho(T-t)} \sigma(x_t, u_t) dw_t,$$

then take expectations on both sides to get

$$\mathbb{E}[x_T] = e^{-\rho T} x + \int_0^T e^{-\rho(T-t)} \mathbb{E}[u_t] dt,$$

where the interchange of the order of integration follows by Fubini’s theorem using the assumption $u \geq 0$. It is now easy to guess that the functional form of σ will not influence the optimal advertising strategy, which is expected to be of the bang-bang type. This is made precise in what follows. From the managerial point of view, this means that the firm, independently of the intensity of the noise and of its dependence on the level of goodwill and rate of advertising spending, will do its best in terms of maximizing *expected* goodwill by simply concentrating all its advertising efforts in a specific period of the advertising campaign.

Assuming $U = [0, m]$, the Bellman equation associated to the problem of maximizing v^u over \mathfrak{U} is given by

$$\psi_t + \sup_{u \in [0, m]} (L^u \psi - e^{-ct} u) = 0, \quad \psi(T, x) = \gamma x. \quad (11)$$

Note that one has

$$\sup_{u \in [0, m]} (L^u \psi - e^{-ct} u) = \begin{cases} -\rho x \psi_x + \frac{1}{2} \sigma^2 \psi_{xx}, & \psi_x \leq e^{-ct} \\ -\rho x \psi_x + m(\psi_x - e^{-ct}) + \frac{1}{2} \sigma^2 \psi_{xx}, & \psi_x > e^{-ct}. \end{cases}$$

Let us consider first the case $\psi_x > e^{-ct}$. Then (11) can be written as

$$\psi_t - (\rho x - m) \psi_x + \frac{1}{2} \sigma^2 \psi_{xx} - m e^{-ct} = 0, \quad \psi(T, x) = \gamma x.$$

We guess a solution of the form $\psi(t, x) = \gamma(t)x + b_1(t)$, obtaining

$$x\gamma'(t) + b_1'(t) - (\rho x - m)\gamma(t) - m e^{-ct} = 0,$$

with terminal conditions $\gamma(T) = \gamma$, $b_1(T) = 0$. Then this equation splits into

$$\gamma'(t) - \rho\gamma(t) = 0, \quad \gamma(T) = \gamma,$$

with solution $\gamma(t) = \gamma e^{-\rho(T-t)}$, and

$$b_1'(t) = -m\gamma(t) + m e^{-ct}, \quad b_1(T) = 0,$$

with solution

$$b_1(t) = -\frac{m\gamma}{\rho}(1 - e^{-\rho(T-t)}) + \frac{m}{c}(e^{-cT} - e^{-ct}).$$

The case $\psi_x \leq e^{-ct}$ is completely similar: let t_* be the solution of the equation $\gamma(t) = e^{-ct}$, i.e. $t_* = \frac{\rho T - \log \gamma}{\rho + c}$. Let us now solve the equation

$$\psi_t - \rho x \psi_x + \frac{1}{2} \sigma^2 \psi_{xx} = 0, \quad \psi(t_*, x) = \gamma(t_*)x + b_1(t_*),$$

where the terminal condition is such that a global solution of (11) equation is at least continuous. It is immediate that the solution of this equation is $\psi(t, x) = \gamma(t)x + b_1(t_*)$, so that the global solution of (11) is $\psi(t, x) = \gamma(t)x + b(t)$, where $b(t) = b_1(t_*)$ for $\gamma(t) \leq e^{-ct}$, and $b(t) = b_1(t)$ for $\gamma(t) > e^{-ct}$. It is also easy to see that b is continuously differentiable on $(0, T)$. In fact, one only needs to check whether there is smooth fit at t_* . But since $b'_1(t) = -m\gamma(t) + me^{-ct}$, by definition of t_* it immediately follows $b'_1(t_*) = 0$. This also proves that $\psi \in C^{1,2}([0, T], \mathbb{R})$, hence the solution of the Bellman equation (11) is the value function of the corresponding control problem, and we can conclude that the optimal control is given by the following bang-bang policy:

$$u_*(t) = \begin{cases} 0 & t \leq t_*, \\ m & t > t_*. \end{cases} \quad (12)$$

That is, it is optimal not to advertise until a certain point in time t_* , after which it becomes optimal to advertise at the maximum rate. Note that, depending on γ , it could well be that $t_* > T$, i.e. it would never be optimal to advertise. This situation arises if the reward for improving the image of a product is small compared to the value of resources spent on advertisement.

We collect the findings of this section in the following proposition.

Proposition 2 *The optimal control problem of maximizing $v^u(0, x)$ is solved by a control of the type (12), with $t_* = \frac{\rho T - \log \gamma}{\rho + c}$, and the corresponding value function is given by $v(t, x) = \gamma(t)x + b(t)$, where*

$$b(t) = \begin{cases} b_1(t_*) & t \leq t_*, \\ b_1(t) & t > t_*. \end{cases}$$

Using a Lagrange multiplier method, we can treat the related problem of maximizing the level of goodwill at time T with a certain available budget for advertising. More precisely, let us consider the constrained stochastic control problem

$$\sup_{u \in \mathcal{M}} \mathbb{E}[x_T], \quad (13)$$

where $\mathcal{M} \subset \mathcal{U}$ is the set of admissible controls $u(\cdot) \in [0, m]$ satisfying the integral constraint

$$\mathbb{E} \left[\int_0^T e^{-ct} u_t dt \right] \leq M,$$

where M is a fixed positive constant. In order for the constrained problem to be non-trivial, it is also necessary to assume that $M \leq m \int_0^T e^{-ct} dt$. We actually only need to consider controls u for which the constraint is binding, i.e. advertising policies that use the whole budget M . In fact, denoting by x_T^u the controlled goodwill at time T , it is clear that $u_1 \geq u_2$ implies $\mathbb{E}[x_T^{u_1}] \geq \mathbb{E}[x_T^{u_2}]$, so it is never optimal to leave resources unused.

Let us introduce a Lagrange multiplier $\lambda > 0$, and consider the (unconstrained) problem

$$\sup_{u \in [0, m]} \mathbb{E} \left[x_T - \lambda \left(\int_0^T e^{-ct} u_t dt - M \right) \right]. \quad (14)$$

Then one has

$$\begin{aligned} \sup_{u \in \mathcal{M}} \mathbb{E}[x_T] &= \sup_{u \in \mathcal{M}} \mathbb{E} \left[x_T - \lambda \left(\int_0^T e^{-ct} u_t dt - M \right) \right] \\ &\leq \sup_{u \in \mathfrak{U}} \mathbb{E} \left[x_T - \lambda \left(\int_0^T e^{-ct} u_t dt - M \right) \right], \end{aligned}$$

where the first equality comes from the above observation that we only need consider controls that use the whole budget M , and the second inequality is justified by $\mathcal{M} \subseteq \mathfrak{U}$. If the unconstrained problem (14) admits a solution u_λ for all $\lambda > 0$, and a λ_* exists such that $\mathbb{E} \int_0^T e^{-ct} u_{\lambda_*}(t) dt - M = 0$, then $u_* := u_{\lambda_*}$ is an optimal control for the constrained problem. So we proceed to solve

$$\sup_{u \in \mathfrak{U}} \mathbb{E} \left[\frac{1}{\lambda} x_T - \int_0^T e^{-ct} u_t dt \right],$$

whose solution is

$$\begin{aligned} \gamma(t) \leq e^{-ct} &\Rightarrow u_*(t) = 0, \\ \gamma(t) > e^{-ct} &\Rightarrow u_*(t) = m, \end{aligned}$$

with $\gamma(t) = \frac{1}{\lambda} e^{-\rho(T-t)}$.

The starting point for advertisement t_* is given by the solution of the equation $\gamma(t) = e^{-ct}$, so that

$$t_* = \frac{\rho T + \log \lambda}{\rho + c}. \quad (15)$$

We now need to show that $\lambda > 0$ exists such that

$$\int_{t_*}^T m e^{-ct} dt = M. \quad (16)$$

The solution of such an equation is given by

$$\lambda_* = e^{\rho T} \left(c \frac{M}{m} + e^{-cT} \right)^{-\frac{\rho+c}{c}},$$

which is clearly positive. It is now clear how to associate to such a λ_* the optimal solution for the constrained problem. Namely, given λ_* we obtain the optimal switching time t_* by (15), and hence the optimal control as $u_*(t) = m \chi_{\{t > t_*\}}$, where χ is the indicator function.

We have then proved the following result.

Proposition 3 *The optimal advertising policy for the constrained maximization of goodwill (13) is given by*

$$u_*(t) = \begin{cases} 0 & t \leq t_*, \\ m & t > t_*, \end{cases}$$

with

$$t_* = \frac{2\rho}{\rho + c} T - \frac{1}{c} (e^{-cT} + cM/m).$$

Remark 4 It follows from (16) that the time to start advertising is given by

$$e^{-ct_*} - e^{-cT} = c \frac{M}{m},$$

and therefore we cannot simply consider unbounded controls with cumulative discounted cost less or equal than M , otherwise the optimal policy would be to “do infinite advertising at time T ”.

4 Quadratic reward and loss functions

In this section we assume that both φ and h are quadratic. In particular, we assume $\varphi(x) = \gamma_0 x^2$, with $\gamma_0 > 0$, and $h(x) = x^2$. That is, we consider the problem of characterizing

$$v(s, x) = \sup_{u \in \mathfrak{U}} \mathbb{E}_{s,x}^u \left[\gamma x_T^2 - \int_0^T e^{-ct} u_t^2 dt \right],$$

or, equivalently,

$$v(s, x) = \inf_{u \in \mathfrak{U}} \mathbb{E}_{s,x}^u \left[\int_0^T e^{-ct} u_t^2 dt - \gamma x_T^2 \right], \quad (17)$$

where we set $\gamma = e^{-cT} \gamma_0$, x_t follows the controlled dynamics

$$x_t = x + \int_s^t (-\rho x_r + u_r) dr + \int_s^t (\sigma_1 x_r + \sigma_2 u_r) dw_r,$$

and \mathfrak{U} is the set of adapted, nonnegative, square integrable controls. A peculiar feature of the problem is that, while the choice of the cost function h is rather standard (see e.g. Muller [21]), the reward function φ is convex, hence representative of a risk-seeking firm. Such attitude toward risk could be justified, for instance, by the attempt to profit from the fluctuations of the goodwill level at time T (in fact, note that, grossly speaking, the firm aims at maximizing both the mean and the variance of x_T). Another peculiar feature of $\varphi(x) = x^2$ is that it equally rewards positive and negative goodwill levels at time T . However, we shall show that under our assumptions $x_t \geq 0$ almost surely, hence the symmetry of φ is harmless.

In this section we assume that the intensity of the “background noise” is negligible, so that we can assume $\sigma_0 = 0$. This assumption is essential in order to obtain (meaningful) solutions in closed form.

The problem at hand is a linear quadratic regulator problem with indefinite costs, which can be solved by the methods of Ait Rami, Moore and Zhou [1] (see also Krylov [18]). In particular, the generalized Riccati equation for this problem is

$$\begin{cases} \dot{P} = (2\rho - \sigma_1^2)P + (1 + \sigma_1\sigma_2)^2 \frac{P^2}{e^{-ct} + \sigma_2^2 P}, \\ e^{-ct} + \sigma_2^2 P > 0 \\ P(T) = -\gamma. \end{cases} \quad (18)$$

Recall that one says that (17) is well posed at s if $v(s, x) > -\infty$. By [1], well-posedness of (17) at $s = 0$ is necessary for the global solvability of (18). Under the assumption

$e^{-cT} - \sigma_2^2 \gamma > 0$, one can only ensure that the problem is locally well posed, i.e. that there exists $t_0 < T$ such that the Riccati equation (18) admits a solution in $[t_0, T]$. The following proposition gives explicit sufficient conditions on the data of the problem such that (18) admits a unique global solution on $[0, T]$.

Proposition 5 *Let us define*

$$\begin{aligned} a_1 &= -2\rho - 2\sigma_1\sigma_2^{-1} - \sigma_2^{-2} < 0 \\ a_2 &= 2\rho + \sigma_1^2 + 2\sigma_2^{-2} + 4\sigma_1\sigma_2^{-1} > 0 \\ a_3 &= -(\sigma_1 + \sigma_2^{-1})^2 < 0 \\ a_4 &= -\gamma_0\sigma_2^2 + 1 > 0 \\ \zeta &= a_2^2 - 4a_1a_3 = 4\rho^2 + \sigma_1^4 - 4\rho\sigma_1^2, \end{aligned}$$

where the inequality $a_4 > 0$ is taken as an assumption. Then the following hold:

- (i) If $\zeta > 0$ and $a_2 > (2|a_1|a_4 - \zeta^{1/2})^+$, then the problem is well posed.
- (ii) If $\zeta > 0$ and $a_2 \leq (2|a_1|a_4 - \zeta^{1/2})^+$, then the problem is well posed if and only if

$$T \leq \zeta^{-1/2} \left(\xi_1 \log \frac{\xi_1}{\xi_1 - a_4} - \xi_2 \log \frac{\xi_2}{\xi_2 - a_4} \right),$$

where $\xi_{1,2} = (-a_2 \pm \zeta^{1/2})/(2a_1)$.

- (iii) If $\zeta = 0$ and $a_2 > 2|a_1|a_4$, then the problem is well posed.
- (iv) If $\zeta = 0$ and $a_2 \leq 2|a_1|a_4$, then the problem is well posed if and only if

$$T \leq a_1^{-1} \left(\log \frac{a_2}{2a_1a_4 + a_2} + \frac{2a_1a_4}{2a_1a_4 + a_2} \right)$$

- (v) If $\zeta < 0$, then the problem is well posed if and only if

$$\begin{aligned} T \leq (2a_1)^{-1} & \left(\log \frac{a_3}{a_1a_4^2 + a_2a_4 + a_3} \right. \\ & \left. - 2a_2\zeta^{-1/2} \left(\operatorname{atan} a_2\zeta^{-1/2} - \operatorname{atan} \zeta^{-1/2}(2a_1a_4 + a_2) \right) \right) \end{aligned}$$

Proof. We shall divide the proof in three steps. In the first step we introduce an auxiliary LQ problem whose well-posedness is sufficient for the well-posedness of (17). In the second step we rescale the auxiliary LQ problem, and in the third and last step we study the global solvability of its associated Riccati equation.

STEP 1: Let us prove that if

$$\inf_{u \in \mathfrak{U}} \mathbb{E} \left[\int_0^T u_t^2 dt - \gamma_0 x_T^2 \right] > -\infty, \quad (19)$$

then (17) is well posed. In fact, suppose, by contradiction, that there exists a sequence $u(k) \in \mathfrak{U}$ such that

$$\mathbb{E}^{u(k)} \left[\int_0^T e^{-ct} u_t(k)^2 dt - \gamma x_T^2 \right] \rightarrow -\infty.$$

Then one also has

$$e^{-cT} \mathbb{E}^{u(k)} \left[\int_0^T u_t(k)^2 dt - \gamma_0 x_T^2 \right] \leq \mathbb{E}^{u(k)} \left[\int_0^T e^{-ct} u_t(k)^2 dt - \gamma x_T^2 \right] \rightarrow -\infty,$$

which contradicts (19), hence our claim is proved.

STEP 2: Let us define $\tilde{x}_t = \sigma_2^{-1} x_t$. Then the minimization problem in (19) is equivalent to

$$\inf_{u \in \mathfrak{U}} \mathbb{E} \left[\int_0^T u_t^2 dt - \gamma_0 \sigma_2^2 \tilde{x}_T^2 \right] \quad (20)$$

subject to

$$\tilde{x}_t = \sigma_2^{-1} x + \int_0^t (-\rho \tilde{x}_s + \sigma_2^{-1} u_s) ds + \int_0^t (\sigma_1 \tilde{x}_s + u_s) dw_s.$$

STEP 3: The Riccati equation for problem (20) is

$$\begin{cases} \dot{P} = (2\rho - \sigma_1^2)P + \frac{(\sigma_1 + \sigma_2^{-1})^2 P^2}{P + 1} \\ P(T) = -\gamma_0 \sigma_2^2 \\ P(t) + 1 > 0, \end{cases}$$

which can be rewritten, after the change of variable $\pi(t) = P(T - t) + 1$, as

$$\begin{cases} \dot{\pi} = a_1 \pi + a_2 + a_3 \pi^{-1} \\ \pi(0) = a_4 \\ \pi(t) > 0. \end{cases} \quad (21)$$

All assertions (i)-(v) are proved simply by determining the first time t_0 for which $\pi(t_0) = 0$. Since the method of proof is the same for all cases, we shall show in detail only cases (i)-(ii). In particular, assuming $\zeta > 0$, then $a_1 \pi^2 + a_2 \pi + a_3 = 0$ has two roots $\xi_1 < \xi_2$, and the reduced Riccati equation (21) as

$$\dot{\pi} = a_1 \pi^{-1} (\pi - \xi_1)(\pi - \xi_2).$$

Note that $a_1 < 0$ implies $\xi_1 \xi_2 = a_3/a_1 > 0$. Moreover one has $\xi_1 + \xi_2 = a_2/|a_1|$, and, by $a_2 > 0$, $0 < \xi_1 < \xi_2$. Therefore $t_0 < \infty$ if and only if $a_4 < \xi_1 = \frac{-a_2 - \zeta^{1/2}}{2a_1}$, which can be rewritten as $a_2 > (2|a_1|a_4 - \zeta^{1/2})^+$. In order to determine t_0 such that $\pi(t_0) = 0$, we shall solve (21) explicitly: for this purpose, note that one has

$$\frac{\pi}{(\pi - \xi_1)(\pi - \xi_2)} = \frac{A}{\pi - \xi_1} + \frac{B}{\pi - \xi_2},$$

with $A = -\frac{\xi_1}{\xi_2 - \xi_1}$ and $B = \frac{\xi_2}{\xi_2 - \xi_1}$, hence (21) can also be written as

$$A \frac{\dot{\pi}}{\pi - \xi_1} + B \frac{\dot{\pi}}{\pi - \xi_2} = a_1.$$

Integrating one gets

$$-\xi_1 \int_0^t \frac{\dot{\pi}(s)}{\pi(s) - \xi_1} ds + \xi_2 \int_0^t \frac{\dot{\pi}(s)}{\pi(s) - \xi_2} ds = a_1 (\xi_2 - \xi_1) t,$$

and finally

$$-\xi_1 \log \frac{\pi(t) - \xi_1}{\pi(0) - \xi_1} + \xi_2 \log \frac{\pi(t) - \xi_2}{\pi(0) - \xi_2} = a_1(\xi_2 - \xi_1)t.$$

Solving for t_0 such that $\pi(t_0) = 0$, recalling that $a_1(\xi_2 - \xi_1) = \zeta^{1/2}$, gives the required result. \square

Assuming that the problem is well posed, the results in [1] imply that the optimal control strategy is unique and is given by the Markov policy

$$u_t^0 = u^0(x_t) = -\frac{(1 + \sigma_1\sigma_2)P(t)}{e^{-ct} + \sigma_2^2 P(t)}x_t, \quad (22)$$

with associated value function $v(s, x) = P(s)x^2$. The optimal trajectory is then given by the closed-loop equation

$$dx_t = a(t)x_t dt + c(t)x_t dw_t,$$

with

$$\begin{aligned} a(t) &:= -\rho - \frac{(1 + \sigma_1\sigma_2)P(t)}{e^{-ct} + \sigma_2^2 P(t)}, \\ c(t) &:= \sigma_1 + \sigma_2 \frac{(1 + \sigma_1\sigma_2)P(t)}{e^{-ct} + \sigma_2^2 P(t)}, \end{aligned}$$

which admits the explicit solution

$$x_t = x \exp \left(\int_0^t (a(s) - \frac{1}{2}c(s)^2) ds + \int_0^t c(s) dw_s \right). \quad (23)$$

In particular, if $x > 0$, then x_t is a.s. positive for all $t \in [0, T]$. If we can prove that $P(t) < 0$ for all $t \in [0, T]$, then (22) will imply that the optimal strategy u_t^0 is positive for all $t \in [0, T]$. The negativity of P is proved in the following lemma.

Lemma 6 *If P solves the Riccati equation (18) on $[s, T]$, then $P(t) < 0$ for all $t \in [s, T]$.*

Proof. Assume, by contradiction, that there exists $t_0 \in [s, T[$ such that $P(t_0) = 0$. Then $P(t) \equiv 0$ is a continuous solution of (18) for $t > t_0$. By uniqueness of the solution of (18) (see [1]) it follows that this is also the only solution. But this contradicts the terminal condition $P(T) = -\gamma \neq 0$. \square

We have thus proved all claims on the optimal state and control.

Remarks. (i) It is worth noting that several sensitivities of the value function and of the expected optimal goodwill with respect to initial data or parameters could be computed as well in terms of the solution of the Riccati equation (18).

(ii) If the intensity of the noise carried by the advertising is also negligible, i.e. we can assume $\sigma_2 = 0$, then the Riccati equation (18) is explicitly solvable, and the corresponding calculations mentioned in the previous remark simplify even more. More details can be found in [19].

(iii) The linear-quadratic regulator approach is also useful if one wants to consider problems with partial observation, which are meaningful in our setting, as goodwill can hardly be measured without error. Suppose instead that we can observe a “noisy proxy” of goodwill z_t specified by

$$\begin{cases} dz_t &= hx_t dt + g dw_t^1 \\ z_0 &= 0, \end{cases} \quad (24)$$

with h and g constants, and w^1 a Brownian motion independent of w . Then one is interested to solve problem (17), where \mathfrak{U} is now the set of nonnegative square integrable controls adapted to the filtration generated by z_t , instead of x_t . Thanks to the so called separation principle (see e.g. Bensoussan [4]), this problem reduces to linear filtering and deterministic control on the filtered dynamics. More details will be given elsewhere.

5 Optimal advertising to meet a goal with discretionary stopping

Problems of mixed optimal stopping and control have recently attracted attention in works of applied probability, see for instance Karatzas and Wang [15] for applications to portfolio optimization, Duckworth and Zervos [9], [10] and Zervos [29] for problems of investment decisions with strategic entry and exit, and Karatzas, Ocone, Wang and Zervos [14] for a singular control problem with finite fuel. For the theory, see, e.g., Krylov [16], Bensoussan and Lions [5], Øksendal and Sulem [23], and Morimoto [20]. One of the first works addressing the issue of finding explicit results was Beneš [3].

In this section we find an explicit representation for the optimal control and the optimal stopping strategy for the case of minimizing an objective function that is the sum of the quadratic distance of the goodwill from a target at the (discretionary) launch time τ and of the cumulative quadratic cost until τ , assuming that the goodwill dynamics is of Ornstein-Uhlenbeck type. For simplicity we also assume $c = 0$, i.e. we consider the case without discounting. In order to discourage long waiting before launching the product, we also introduce an extra term in the objective function depending on the time of launching.

Let $y_t = k - x_t$ be the distance of the goodwill level at time t from a desired target k . We shall find the solution to the problem

$$\inf_{u \in \mathfrak{U}, \tau \in \mathcal{S}} \mathbb{E}_{0,x}^{u,\tau} \left[y^2(\tau) + \gamma_1 \int_0^\tau u^2(t) dt + \gamma_2 \tau \right] =: v(x), \quad (25)$$

where y is such that

$$dy_t = (\mu - \rho y_t + u_t) dt + dw_t,$$

with $\mu := \rho k$ and we have assumed, without loss of generality (in the setting of constant σ), $\sigma = 1$. Let \mathbb{F} be the filtration generated by w . Then \mathfrak{U} is the space of \mathbb{F} -adapted square integrable control processes, and \mathcal{S} is the set of all \mathbb{F} -stopping times.

Note that the first term in (25) assigns equal costs to the events that the target is not reached (from below) and that it is exceeded. Our setting could be considered as a stylized model for the situation when a firm is launching a new product in a

predetermined quantity (the target k) and considers equally undesirable to undersell the product ($x_t < k$) or to leave demand unmet ($x_t > k$).

The quasi-variational inequality associated to the mixed problem of optimal control and optimal stopping (25) can be written as

$$\min_x \left(x^2 - v(x), \min_u (L^u v + \gamma_1 u^2 + \gamma_2) \right) = 0,$$

where L^u is the generator of the controlled diffusion y , i.e. L^u is the differential operator defined by

$$L^u f(x) = \frac{1}{2} f''(x) + (\mu - \rho x - u) f'(x).$$

We guess a continuation region D of the type $D = \{x : x \geq x_0\}$, where one must have

$$\min_u (L^u v + \gamma_1 u^2 + \gamma_2) = 0.$$

We have

$$\min_u (L^u v + \gamma_1 u^2 + \gamma_2) = Av - \frac{1}{4\gamma_1} v_x^2 + \gamma_2,$$

where A is the generator of the uncontrolled diffusion, i.e.

$$Af(x) = \frac{1}{2} f''(x) + (\mu - \rho x) f'(x).$$

Then we get

$$Av - \frac{1}{4\gamma_1} v_x^2 + \gamma_2 = 0, \quad x \geq x_0$$

In order to linearize this ODE, we apply the Hopf-Cole transformation $U(x) = e^{\frac{1}{2\gamma_1} v(x)}$, obtaining

$$\frac{1}{2} U_{xx} + (\mu - \rho x) U_x - \frac{\gamma_2}{2\gamma_1} U = 0, \quad x \geq x_0$$

In order to obtain solutions that are ordinary functions, we restrict ourselves to the special case $\frac{\gamma_2}{2\gamma_1} = \rho$. However, one can solve the linear equation for U without this assumption, obtaining solutions that can be expressed in terms of special functions. Two linearly independent solutions are

$$U_1(x) = e^{-2\mu x + \rho x^2}, \quad U_2(x) = e^{\rho(x - \mu/\rho)^2} \int_x^\infty e^{-\rho(s - \mu/\rho)^2} ds$$

Then the general solution can be written as $U = \alpha_1 U_1 + \alpha_2 U_2$, with α_1 and α_2 arbitrary real constants.

Guided by the observation that U_1 is unbounded in the continuation region, we set $\alpha_1 = 0$, and impose C^1 fit of $U(x) = \alpha_2 U_2(x)$ to the Hopf-Cole transformation of x^2 at the point x_0 , that is

$$\begin{aligned} U(x_0) &= e^{-\frac{1}{2\gamma_1} x_0^2} \\ U'(x_0) &= -\frac{x_0}{\gamma_1} e^{-\frac{1}{2\gamma_1} x_0^2}. \end{aligned}$$

We solve now the following system of equations for the unknowns α_2 and x_0 :

$$\begin{aligned}\alpha_2 e^{\rho(x_0 - \mu/\rho)^2} \int_x^\infty e^{-\rho(s - \mu/\rho)^2} ds &= e^{-\frac{1}{2\gamma_1} x_0^2} \\ \alpha_2 \left[2\rho \left(x_0 - \frac{\mu}{\rho} \right) e^{\rho(x_0 - \mu/\rho)^2} \int_{x_0}^\infty e^{-\rho(s - \mu/\rho)^2} ds - 1 \right] &= -\frac{x_0}{\gamma_1} e^{-\frac{1}{2\gamma_1} x_0^2}.\end{aligned}$$

Therefore x_0 is given by the solution of the following equation

$$\left(\left(2\rho + \frac{1}{\gamma_1} \right) x - 2\mu \right) e^{\rho(x - \mu/\rho)^2} \int_x^\infty e^{-\rho(s - \mu/\rho)^2} ds = 1.$$

In fact the solution, if it exists, is unique, because the left hand side, as a function of x , is increasing. Now one can also find α_2 in terms of x_0 , so that we have a candidate continuation region (or equivalently an optimal stopping region), and a candidate value function.

We need to show that $Av(x) - \frac{1}{4\gamma_1}(v'(x))^2 + \gamma_2 \geq 0$ in the region $x \leq x_0$, for $v(x) = x^2$. That is, we want to show that

$$\left(2\rho + \frac{1}{\gamma_1} \right) x^2 - 2\mu x - (1 + \gamma_2) \leq 0. \quad (26)$$

The expression on the left hand side takes its maximum either at $x = 0$ or at $x = x_0$. Therefore, if $x_{max} = 0$, condition (26) is trivially verified. Otherwise, if $x_{max} = x_0$, we have

$$\frac{1}{(2\rho + \gamma_1^{-1})x_0 - 2\mu} = U_2(x_0) \geq \frac{1}{\rho y_0 + \sqrt{\rho^2 y_0^2 + 2\rho}},$$

where $y_0 := x_0 - \frac{\mu}{\rho}$, and the inequality follows by standard estimates on the error function. Therefore we get

$$(2\rho + \gamma_1^{-1})x_0 - 2\mu \leq \rho y_0 + \sqrt{\rho^2 y_0^2 + 2\rho}.$$

After some algebraic manipulations, one finds

$$\left(2\rho + \frac{1}{\gamma_1} \right) x_0^2 - 2\mu \frac{1}{\gamma_1} x_0 \leq 2\rho\gamma_1 = \gamma_2.$$

Therefore, (26) is verified if, e.g., $\gamma_1 > 1$. We shall assume that in the following, but note that this condition can be weakened.

Finally, we need to prove that $v(x) \geq x^2$ in the continuation region $x \geq x_0$, or equivalently that $U(x) \geq e^{-\frac{1}{2\gamma_1} x^2}$ for $x \geq x_0$. In order to prove this, let us consider their ratio $f(x) = U(x)e^{\frac{1}{2\gamma_1} x^2}$ and prove that it is increasing. One has

$$f'(x) = \alpha_2 e^{\frac{1}{2\gamma_1} x^2} \left(\left(\left(2\rho + \frac{1}{\gamma_1} \right) x - 2\mu \right) U_2(x) - 1 \right),$$

and since we have that $((2\rho + \frac{1}{\gamma_1})x - 2\mu)U_2(x)$ is increasing in x and $((2\rho + \frac{1}{\gamma_1})x_0 - 2\mu)U_2(x) = 1$, it follows $((2\rho + \frac{1}{\gamma_1})x - 2\mu)U_2(x) > 1$ for $x \geq x_0$. Therefore we have verified all conditions for optimality, and we summarize our findings in the following proposition.

Proposition 7 *The optimal control policy u_* and optimal stopping time τ_* for the problem (25), with $\gamma_1 > 1$ and $\frac{\gamma_2}{2\gamma_1} = \rho$, are given by*

$$u_*(y_t) = \arg \min_u (L^u v(y_t) + \gamma_1 u^2 + \gamma_2) = \frac{v'(y_t)}{2\gamma_1}$$

and

$$\tau_* = \inf\{t \geq 0 : y_t \leq x_0\}.$$

6 Further problems

The optimal control problems studied in this paper are limited to the case of “smooth” disturbances, that is, the driving noise process has continuous paths. It is meaningful to relax this assumption and consider also jump components in the noise, to take into account possible shocks to the image of the advertised product, due, for instance, to bad news on the product itself or similar ones, or to the introduction of superior technologies.

One could also try to study different type of controls, namely impulse controls, or even combinations of classical and impulse controls. This is particularly meaningful for our problems, since impulse controls correspond to the so-called “pulsing advertising” policies that have been studied in the management and marketing literature (see [11] and references therein).

Acknowledgements

The author wishes to thank Sergio Albeverio, Victor de la Peña, Fausto Gozzi, Cristian Pasarica, Sergei Savin, and Luciano Tubaro for helpful comments and suggestions on an earlier version of this paper. The comments of two anonymous referees, to whom the author is grateful, considerably improved the presentation of the paper. The financial support of the National Science Foundation under grant DMS-02-05791 (Principal Investigator V. de la Peña) is gratefully acknowledged. Large part of this work was carried out at the Graduate Business School of Columbia University.

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